Ekedahl–Oort Types of some Artin–Schreier Curves

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Background

Artin-Schreier curves are plane curves defined over a field of characteristic p, given by an equation of the form

$$y^p - y = \frac{f(x)}{g(x)}.$$

When $p = 2$, these curves have been studied extensively, as their Jacobian varieties have wide usage throughout cryptography and related subfields. In our project, we study an invariant, called the Ekedahl-Oort type, of curves $C_{p,m}$ given by

$$y^p - y = x^m$$

where $m = 3, 4, 5$ and where m divides $p - 1$.

The Set-Up

Consider an Artin–Schreier curve C of the form

$$y^p - y = f(x)$$

where $f(x)$ is a polynomial of degree m. The *genus* of C is

$$g = \frac{(p-1)(m-1)}{2}.$$

Computing the Ekedahl–Oort (EO)-type of C involves three steps:

- Finding the Hasse–Witt triple.
- Determining the Dieudonn'e module.
- Constructing the final filtration.

Hasse-Witt Triple

The Hasse-Witt Triple associated with C is a g-dimensional vector space Q along with two operators (Φ, Ψ).
For example, for the curve $C_{5,4}$ given by

$$y^5 - y = x^4,$$

Q is defined as

$$\left\langle \frac{y^j}{x^i} : 0 \le j \le p-1, 0 < i < \frac{jm}{p} \right\rangle.$$

For $C_{5,4}$:

$$Q = \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^3}{x^2}, \frac{y^4}{x}, \frac{y^4}{x^2}, \frac{y^4}{x^3} \right\rangle$$

The functions Φ and Ψ are defined from the pth power operator $(-)^p$ on Q, and Ψ is only defined on the kernel of Φ. In our example:

q	y^2/x	y^3/x	y^4/x	y^3/x^2	y^4/x^2	y^4/x^3
$\Phi(q)$	0	$3y^2/x$	$4y^3/x$	0	0	0
$\Psi(q)$	$3(y^4/x^3)^\vee$	$-$	$-$	$2(y^4/x^2)^\vee$	$3(y^3/x^2)^\vee$	$(y^4/x)^\vee$

Dieudonné Module

We need to find the Dieudonné module associated with this curve. This consists of a $2g$ dimensional vector space M and two semi-linear operators from M to itself (F, V). For $C_{5,4}$, M is

$$M = \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^3}{x^2}, \frac{y^4}{x}, \frac{y^4}{x^2}, \frac{y^4}{x^3} \right\rangle \oplus \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^3}{x^2}, \frac{y^4}{x}, \frac{y^4}{x^2}, \frac{y^4}{x^3} \right\rangle^\vee.$$

The function F is defined on Q, sending everything in the kernel of Φ to $\Psi(Q)$, and everything else to $\Phi(Q)$. In our example this is:

q	y^2/x	y^3/x	y^4/x	y^3/x^2	y^4/x^2	y^4/x^3
$F(q)$	$3(y^4/x^3)^\vee$	$3y^2/x$	$4y^3/x$	$2(y^4/x^2)^\vee$	$3(y^3/x^2)^\vee$	$(y^4/x)^\vee$

The function V is defined on M using $\Phi^\vee$ and $\Psi^\vee$. In our example:

q	y^2/x	y^3/x	y^4/x	y^3/x^2	y^4/x^2	y^4/x^3
$V(q)$	0	0	$4(y^4/x^3)^\vee$	$2(y^4/x^2)^\vee$	$3(y^3/x^2)^\vee$	$2(y^2/x)^\vee$

and

λ	$(y^2/x)^\vee$	$(y^3/x)^\vee$	$(y^4/x)^\vee$	$(y^3/x^2)^\vee$	$(y^4/x^2)^\vee$	$(y^4/x^3)^\vee$
$V(\lambda)$	$3(y^3/x)^\vee$	$4(y^4/x)^\vee$	0	0	0	0

Canonical Filtration

The canonical filtration is the coarsest filtration of $V(M)$ which is stable under V and F^{-1}. We want to find

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq M_4 \subseteq M_5 \subseteq M_6.$$

For the curve $C_{5,4}$ we have:

$$M_6 = V(M) = \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^4}{x}, \frac{y^3}{x^2}, \frac{y^4}{x^2}, \frac{y^4}{x^3} \right\rangle^\vee,$$

$$M_5 = V(F^{-1}(V(M))) = \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^4}{x}, \frac{y^3}{x^2}, \frac{y^4}{x^2} \right\rangle^\vee,$$

$$M_3 = V(F^{-1}(V^2(M))) = \left\langle \frac{y^2}{x}, \frac{y^3}{x}, \frac{y^4}{x} \right\rangle^\vee,$$

$$M_2 = V^2(M) = \left\langle \frac{y^3}{x}, \frac{y^4}{x} \right\rangle^\vee, M_1 = V^3(M) = \left\langle \frac{y^4}{x} \right\rangle^\vee.$$

Computing the EO type

We have our canonical filtration

$$V^3(M) \subset V^2(M) \subset V(F^{-1}(V^2(M))) \subset V(F^{-1}(V(M))) \subset V(M).$$

The EO type is the sequence $[\dim(V(M_i))]$. While there is no M_4 in the canonical filtration, we can fill in this gap by picking either element as both go to 0 under V. In $C_{5,4}$ the Ekedahl-Oort type is:

$$[0, 1, 2, 2, 2, 2].$$

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Data

As a part of this project we wrote a program to compute the EO-type of curves of the form $y^p - y = x^m$. Some example calculations:

(p,m)	Ekedahl-Oort Type
(7, 3)	[0, 0, 1, 2, 2, 2]
(13, 3)	[0, 0, 0, 0, 1, 2, 3, 4, 4, 4, 4, 4]
(5, 4)	[0, 1, 2, 2, 2, 2]
(13, 4)	[0, 0, 0, 1, 2, 3, 4, 5, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6]
(11, 5)	[0, 0, 1, 2, 3, 4, 4, 4, 5, 6, 6, 6, 7, 8, 8, 8, 8, 8, 8, 8]

Table: EO-Types of specific curves

Theorems

Suppose $p \equiv 1 \pmod{m}$. Let C be an Artin–Schreier curve with affine equation $y^p - y = x^m$. Then the Ekedahl–Oort type of C for $m = 3$ is:

$$[\underbrace{0, 0, \ldots, 0}_{(p-1)/3\text{-times}}, 1, 2, 3, \ldots, \frac{p-4}{3}, \underbrace{\frac{p-1}{3}, \frac{p-1}{3}}_{(p+2)/3\text{-times}}]$$

For $m = 4$ is:

$$[\underbrace{0, 0, \ldots, 0}_{(p-1)/4\text{-times}}, 1, 2, 3, \ldots, \frac{p-3}{2}, \underbrace{\frac{p-1}{2}, \frac{p-1}{2}}_{(3p+1)/4\text{-times}}]$$

and for $m = 5$ is:

$$[\underbrace{0, \ldots, 0}_{(p-1)/5\text{-times}} \; 1, 2, \ldots, \underbrace{\frac{2p-2}{5}, \ldots, \frac{2p-2}{5}}_{(p+4)/5\text{-times}}, \frac{2p+3}{5}, \ldots,$$
$$\underbrace{\frac{3p-3}{5}, \ldots \frac{3p-3}{5}}_{(p+4)/5\text{-times}}, \frac{3p+2}{5}, \ldots \underbrace{\frac{4p-4}{5}, \ldots, \frac{4p-4}{5}}_{(3p+2)/5\text{-times}},]$$

Future Directions

From our data, we observe that, for a fixed m, there is a pattern in the EO-type of the curve $C_{p,m}$ as p varies. Questions:

- What is the general form of the EO type of the curve $C_{p,m}$?
- Does the EO type remain the same if x^m is replaced by a another polynomial of degree m?

References

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Acknowledgments

A special thanks to Skidmore College's Faculty Student Summer Research Program, and to our research advisor, Patrick Daniels.